

ON CENTERS OF BLOCKS WITH NON-CYCLIC DEFECT GROUPS

YOSHIHIRO OTOKITA

ABSTRACT. In this short note we study the center ZB of a block B of a finite group over an algebraically closed field of prime characteristic through its Loewy length $\text{ll}ZB$. A result of Okuyama in 1981 gave an upper bound for $\text{ll}ZB$ in terms of defect group of B . The purpose of this note is to improve this bound for non-cyclic defect groups.

1. INTRODUCTION

In this short note we study the center of a block of a finite group over an algebraically closed field of prime characteristic through its Loewy length.

Let G be a finite group and F an algebraically closed field of characteristic $p > 0$. For a block B of the group algebra FG we denote by ZB its center. In order to examine the structure of ZB we use its Loewy length $\text{ll}ZB$, that is, the nilpotency index of the Jacobson radical JZB . A result of Okuyama [4] states that $\text{ll}ZB \leq p^d$ where d is the defect of B . In addition Koshitani-Külshammer-Sambale [2] determines $\text{ll}ZB$ for cyclic defect groups. By this, we consider the other cases in this note. More precisely, we prove the following theorem:

Theorem 1. *Let B be a block of FG with non-cyclic defect group of order p^d . Then*

$$\text{ll}ZB \leq p^{d-1} + p - 1.$$

Now let us take a block B with defect group D of order 3^5 as an example. Then $\text{ll}ZB \leq 243$ by Okuyama's formula. Theorem 1 implies that D is cyclic provided $84 \leq \text{ll}ZB \leq 243$. In this case $\text{ll}ZB = 243$ or 122 by [2]. In all other cases we have $\text{ll}ZB \leq 83$.

We remark that the converse of this theorem is not true in general. For instance a block B with cyclic defect group C_{p^2} and inertial quotient group C_{p-1} satisfies $\text{ll}ZB = p + 2 \leq 2p - 1$ whenever $p \geq 3$.

2. PRELIMINARIES

We prepare some notations. For a conjugacy class $C \in \text{Cl}(G)$ its defect group $\delta(C)$ is defined as a Sylow p -subgroup of $C_G(x)$ where $x \in C$. For a p -subgroup P of G we set

$$I_G(P) = \sum_{C \in \text{Cl}(G), \delta(C) \leq_G P} FC^+, \quad \tilde{I}_G(P) = \sum_{C \in \text{Cl}(G), \delta(C) <_G P} FC^+$$

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where C^+ is the class sum of C . These are ideals of the center ZFG of FG . Furthermore we denote by $t(P)$ the Loewy length of FP following Wallace [9]. Here we refine a lemma in Passman [7].

Lemma 2. *Let P be a p -subgroup of G . Then the following hold:*

- (1) $I_G(P) \cdot JZFG^{t(Z(P))} \subseteq \tilde{I}_G(P)$.
- (2) $I_G(P) \cdot JZFG^{(p^{a+1}-1)/(p-1)} = 0$ where $|P| = p^a$.

Proof. It remains only to prove (1) by [7, Lemma 3 (ii)]. Let $\text{Br}_P : ZFG \rightarrow ZFC_G(P)$ be the Brauer homomorphism associated to P . Since Br_P maps nilpotent elements to nilpotent elements we have $\text{Br}_P(JZFG) \subseteq JZFC_G(P)$. On the other hand $\text{Br}_P(I_G(P)) \subseteq I_{C_G(P)}(Z(P))$ holds (see the proof of [7, Lemma 3 (i)]). Thus it follows from [7, Lemma 2 (i)] that

$$\begin{aligned} \text{Br}_P(I_G(P) \cdot JZFG^{t(Z(P))}) &\subseteq I_{C_G(P)}(Z(P)) \cdot JZFC_G(P)^{t(Z(P))} \\ &= JFZ(P)^{t(Z(P))} \cdot I_{C_G(P)}(Z(P)) = 0 \end{aligned}$$

since $Z(P)$ is central in $C_G(P)$. Therefore we deduce

$$I_G(P) \cdot JZFG^{t(Z(P))} \subseteq \text{KerBr}_P \cap I_G(P) = \tilde{I}_G(P)$$

as claimed. \square

3. PROOF OF MAIN THEOREM

We first improve Külshammer-Sambale [3, Theorem 12 and Proposition 15] by using Lemma 2.

Proposition 3. *Let B be a block of FG with non-abelian defect group of order p^d . Then $\text{ll}ZB < p^{d-1}$.*

Proof. We may assume $p \neq 2$ by [3, Proposition 15]. Let D be a defect group of B . If $Z(D)$ is cyclic of order p^{d-2} then D is one of the following types:

$$\begin{aligned} M_{p^d} &:= \langle x, y \mid x^{p^{d-1}} = y^p = 1, y^{-1}xy = x^{p^{d-2}+1} \rangle, \\ W_{p^d} &:= \langle x, y, z \mid x^{p^{d-2}} = y^p = z^p = [x, y] = [x, z] = 1, [y, z] = x^{p^{d-3}} \rangle \end{aligned}$$

where $d \geq 3$. In both cases $\text{ll}ZB < p^{d-1}$ (see [3, Proposition 10 and Lemma 11]). If D has a cyclic subgroup of index p then $D \simeq M_{p^d}$ (e.g. see [1, Chapter 5, Theorem 4.4]). Thereby we need only consider the other cases. Since $Z(D)$ is non-cyclic or has order at most p^{d-3} , we have $\lambda_0 := p^{d-3} + p - 1 \geq t(Z(D))$. Thus we first obtain from Lemma 2 (1) that

$$I_G(D) \cdot JZFG^{\lambda_0} \subseteq I_G(D) \cdot JZFG^{t(Z(D))} \subseteq \tilde{I}_G(D) = \sum_{D_1 < D} I_G(D_1).$$

By our assumptions above, D_1 is non-cyclic or has order at most p^{d-2} . In both cases we have $\lambda_1 := p^{d-2} + p - 1 \geq t(Z(D_1))$. Thus

$$\begin{aligned} I_G(D) \cdot JZFG^{\lambda_0 + \lambda_1} &\subseteq \sum_{D_1 < D} I_G(D_1) \cdot JZFG^{\lambda_1} \subseteq \sum_{D_1 < D} I_G(D_1) \cdot JZFG^{t(Z(D_1))} \\ &\subseteq \sum_{D_1 < D} \tilde{I}_G(D_1) = \sum_{D_2 < D_1 < D} I_G(D_2). \end{aligned}$$

Finally, it follows from Lemma 2 (2) that

$$I_G(D) \cdot JZFG^{\lambda_0+\lambda_1+\lambda_2} \subseteq \sum_{D_2 < D_1 < D} I_G(D_2) \cdot JZFG^{\lambda_2} = 0$$

where $\lambda_2 := p^{d-1} - 1/p - 1$ since $|D_2| \leq p^{d-2}$. Now let e be the block idempotent of B . Then

$$JZB^{\lambda_0+\lambda_1+\lambda_2} = eJZFG^{\lambda_0+\lambda_1+\lambda_2} \subseteq I_G(D) \cdot JZFG^{\lambda_0+\lambda_1+\lambda_2} = 0$$

and this means $\text{ll}ZB \leq \lambda_0 + \lambda_1 + \lambda_2$. Accordingly, $\text{ll}ZB < p^{d-1}$ except for one case that $p = 3$ and $d = 4$. Hence we consider this case in the following. From [6] (see [5, proof of Theorem 1.3]), there exists a non-trivial B -subsection (u, b) such that

$$\text{ll}ZB \leq (|u| - 1)\text{ll}Z\bar{b} + 1$$

where \bar{b} is the unique block of $F[C_G(u)/\langle u \rangle]$ dominated by b . We may assume that \bar{b} has defect group $C_D(u)/\langle u \rangle$ by Sambale [8, Lemma 1.34]. We put $|u| = 3^s$ and $|C_D(u)| = 3^r$. If $r \leq d - 2$ then

$$\text{ll}ZB \leq (3^s - 1)3^{r-s} + 1 \leq (3^s - 1)3^{d-s-2} + 1 < 3^{d-1}.$$

In case of $r = d - 1$, we may $r > s$ by our assumptions and thus

$$\text{ll}ZB \leq (3^s - 1)3^{r-s} + 1 = 3^r - 3^{r-s} + 1 < 3^r = 3^{d-1}$$

as required. Therefore we may assume $d = r$, so that $u \in Z(D)$. Hence $|u| = 3$ and $D/\langle u \rangle$ is isomorphic to $C_3 \times C_3 \times C_3, C_9 \times C_3, M_{27}$ or W_{27} by our assumptions. In all cases $\text{ll}Z\bar{b} \leq 11$ by [3, Theorem 1, Proposition 10 and Lemma 11]. Consequently, $\text{ll}ZB \leq 23 < 27 = p^{d-1}$. Our claim is completely proved. \square

Theorem 1 is an immediate corollary to Proposition 3.

Proof of Theorem 1. We may assume $p^{d-1} < \text{ll}ZB$ and thus D is abelian by Proposition 3. In this case Külshammer-Sambale [3] has proved that $\text{ll}ZB \leq t(D)$. Hence our claim follows. \square

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YOSHIHIRO OTOKITA:

DEPARTMENT OF MATHEMATICS AND INFORMATICS
GRADUATE SCHOOL OF SCIENCE
CHIBA UNIVERSITY

E-mail address: otokita@chiba-u.jp